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SMOOTHING FOR SPECTRAL METHODS

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Abstract

The purpose of this article is mainly to demonstrate, by means of simple examples, that some kind of smoothing must be an essential part of any spectral method.

Spectral methods have, in principle, infinite order of accuracy, if the true solution is smooth. I shall review some old work, done jointly with Majda and McDonough [1], which shows how drastically the situation changes when discontinuities are present. The error pollutes the solution globally if no smoothing is used, and pollutes it in a very large region, even if smoothing based on the finite spectral transform is used. A more drastic type of smoothing will remove this error.

Smoothing is, in general, necessary for two reasons:

a. Accuracy

b. Stability

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1. Error Analysis for Smooth Solutions

I shall begin with a periodic one-dimensional very simple problem:

$$u_t = au_x, \tag{1.1}$$

to be solved for $-\pi < x < \pi$, $0 < t$, with a, some fixed constant, and initial data

$$u(x,0) = \phi(x).$$

The Fourier method proceeds by breaking up the x interval via

$$x_\nu = \nu h, \qquad \nu = 0, \pm 1, \ldots, \pm N,$$

with $(2N+1)h = 2\pi$. For any $u(x)$, one finds its trigonometric interpolant

$$u^h(x) = (I^h u)(x) \; ,$$

as follows. First one computes the finite Fourier transform:

$$(Fu)(j) = \tilde{u}(j) = 1/(2N+1) \sum_{\nu=-N}^{N} e^{-ijx_\nu} u(x_\nu)$$

(1.2)

$$j = 0, \pm 1, \ldots, \pm N.$$

Then one obtains the interpolant:

$$u^h(x) = F^{-1}(\hat{u}) = \sum_{j=-N}^{N} e^{ijx} \tilde{u}(j). \tag{1.3}$$

It follows that

$$u^h(x_\nu) = u(x_\nu) \quad \text{for} \quad \nu=0,\pm1,\ldots,\pm N.$$

The approximate solution to the Cauchy problem (1.1) could be obtained by solving exactly the finite dimensional problem:

$$\frac{\partial u^h}{\partial t} = a\,\frac{\partial u^h}{\partial x} \tag{1.4}$$

$$u^h(x,0) = \phi^h(x).$$

The solution is

$$u^h(x,t) = \sum_{j=-N}^{N} \tilde{\Phi}(j)e^{aijt+ijx}. \tag{1.5}$$

To find the <u>true</u> solution one needs the <u>true</u> Fourier coefficients:

$$\hat{u}(j) = 1/2\pi \int_{-\pi}^{\pi} e^{-ijx}u(x)dx. \tag{1.6}$$

One then has:

$$u(x) = \sum_{j=-\infty}^{\infty} \hat{u}(j)e^{ijx}. \tag{1.7}$$

For (1.1), the true solution is thus the infinite sum:

$$u(x,t) = \sum_{j=-\infty}^{\infty} \hat{\phi}(j)e^{aijt+ijx}. \tag{1.8}$$

Thus we have a simple expression for the error:

$$u - u^h = \sum_{|j|>N} \hat{\phi}(j) e^{aijt+ijx} \tag{1.9}$$

$$+ \sum_{|j|\le N} [\hat{\phi}(j) - \tilde{\phi}(j)] e^{aijt+ijx}$$

$$= E_I + E_{II}.$$

The relationship between $\hat{\phi}(j)$, the true Fourier coefficients, and $\tilde{\phi}(j)$, the finite Fourier coefficient is easily seen to be

$$\tilde{\phi}(j) = \sum_{\mu=-\infty}^{\infty} \hat{\phi}\big(j+\mu(2N+1)\big). \tag{1.10}$$

See Kreiss-Oliger [3].

Thus, we may rewrite (1.9):

$$u - u^h = \sum_{|j|>N} \hat{\phi}(j) e^{aijt + ijx} \tag{1.11}$$

$$+ \sum_{|j|\le N} e^{aijt + ijx} \Big(\sum_{\substack{\mu=-\infty \\ \mu\neq 1}}^{\infty} \hat{\phi}(j+\mu(2N+1)) \Big).$$

Now if $\phi(x)$ is a smooth function, i.e., $\phi(x) \in C^{\infty}$, then for any $K>0$, there exists universal constants C_k, so that

$$|\hat{\phi}(j)| < C_K (1+|j|)^{-K}.$$

This, together with (1.11), implies

$$\|u-u^h\| < C_p h^p, \tag{1.12}$$

in any reasonable norm, as $p \downarrow 0$. (We use the convention that universal constants are always denoted $C_{\text{subscript}}$). This explains the "infinite" order accuracy.

2. Deterioration Due to Discontinuities

Suppose, on the other hand, ϕ is not smooth at $x = 0$. As an example, we take

$$\phi(x) = |x|^\gamma \, \sigma(x), \ \gamma > 0, \tag{2.1}$$

with $0 \le \sigma \le 1$, a smooth function, $\sigma \equiv 1$ if $|x| < \pi - 2\varepsilon$, $\sigma \equiv 0$ if $|x| > \pi - \varepsilon$, for small $\varepsilon > 0$. Then $\hat{\phi}(j)$ satisfies

$$\left|\left(\frac{\partial}{\partial j}\right)^\alpha \hat{\phi}(j)\right| = 0\left((1+|j|)^{-1-\gamma-\alpha}\right) .$$

The approximate solution has a <u>global</u> error. Let R_δ be a region of smoothness for the exact solution

$$R_\delta = \{(x,t)/|x+at| > \delta > 0\}.$$

In [1], we show the following global error estimate.

<u>Lemma 1</u>.

(1) $$\max_{(x,t)\,\varepsilon\, R_\delta} |u(x,t) - u^h(x,t)| \le C_\delta \, h^{1+\gamma} ,$$

(2) $$\lim_{h\downarrow 0} \max_{(x,t)\,\varepsilon\, R_\delta} \left(\frac{|u(x,t) - u^h(x,t)|}{h^{1+\delta}}\right) > C > 0 .$$

For a Heaviside function type of initial data:

$$\phi(x) = (\operatorname{sgn} x)\ \rho(x),$$

with $\phi(0) = 0$, the global error is $O(h^2)$. This anamoly occurs because of a cancellation in the finite Fourier series for this function. If we took any nonzero value for $\phi(0)$, the error would be $O(h)$, as expected.

3. A Simple and Inadequate Smoothing Technique

For smooth $\phi(x)$ defined as in the previous section, we filter the initial data and use

$$u^h(x,0) = \sum_{j=-N}^{N} \phi(jh)\ e^{ijx}\tilde{\phi}(j). \tag{3.1}$$

Note, we are still using the finite Fourier transform of the initial data. Then

$$u - u^h = \sum_{j=-\infty}^{\infty} e^{ij(x+at)}\big(1-\sigma(jh)\big)\hat{\phi}(j) \tag{3.2}$$

$$- \sum_{|j|<N} e^{ij(x+at)}\phi(jh) \sum_{|\nu|\geq 1} \hat{\phi}\big(j+\nu(2N+1)\big) = E_I + E_{II}.$$

It is shown in [1] that

$$\max_{(x,t)\ \in\ R_\delta} |E_I| < C_\lambda h^\lambda.$$

for any $\lambda > 0$.

If $\phi(x)$ is the Heaviside function, then we also show that, in

R_δ, E_{II} is within $O(h^\lambda)$, for any λ, of the solution to the true Cauchy problem (1.1), with initial data:

$$u(x,0) = h^2[C_1\delta'(0) + hC_2\delta''(0) + \ldots] \ . \tag{3.3}$$

Here $\delta(x)$ is the Dirac delta function, and δ' is its distribution derivative, etc.

Since there is no coupling for the problem (1.1), the support of this distribution does not spread into R_δ when we solve the problem. Thus for this decoupled case we luckily have infinite order accuracy in R_δ. But this result is false in more complicated cases.

Suppose, for example, we consider the coupled hyperbolic system

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \tag{3.4}$$

with initial data

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \end{pmatrix} ,$$

for ϕ the Heaviside function. Then, there is a global deterioration of accuracy within the range of influence of the orgin.

Let

$$R_\delta^{(1)} = \{(x,t)\,|\,-1 + \delta < \frac{x}{t} < 1-\delta\} \ .$$

Then the analogous expressions for E_I and E_{II} yield the results

$$|E_I| = O(h^\lambda) \quad \text{for all} \quad \lambda > 0 \quad \text{in} \quad R_\delta^{(1)}.$$

However, E_{II} is, within $O(h^3)$, the solution of the Cauchy problem (3.4) with initial data

$$u(x,0) = h^2 C_1 \begin{bmatrix} \delta'(0) \\ 0 \end{bmatrix}, \quad C_1 \neq 0. \tag{3.5}$$

Because of the coupling through the lower order terms, the entire shadow region, $R_\delta^{(1)}$, feels the influence of this initial data and

$$\overline{\lim_{(x,t)\,\varepsilon\, R_\delta^{(1)}}} \frac{|E_{II}|}{h^2} > C .$$

Thus, we have a global deterioration of accuracy.

In earlier joint work with Majda [2], we analyzed this large error phenomenon for dissipative finite-difference schemes. Since such (usually simpler) methods do not strive for infinite order accuracy, their comparable deterioration might not trouble the finite-difference user, as much as it might disturb the proponent of spectral methods.

4. A Drastic Smoothing Technique

For constant coefficient linear hyperbolic problems, we can remove the large region of low accuracy by using and smoothing the true Fourier coefficients of the initial data. Let

$$u^h(x,0) = \sum_{j=-N}^{N} \sigma(jh)\, e^{ijx}\, \hat{\phi}(j) \tag{4.1}$$

Actually, if $\phi(x) = \phi^{(1)} + \phi^{(2)}$, with $\phi^{(1)}$ discontinuous and $\phi^{(2)}$ smooth, we can take if convenient:

$$u^h(x,0) = \sum_{j=-N}^{N} \sigma(jh)e^{ijx}\,\hat{\phi}^{(1)}(j) + \sum_{j=-N}^{N} \sigma(jh)e^{ijx}\,\tilde{\phi}^{(2)}(j)]. \tag{4.2}$$

We then get infinite order of accuracy in regions where the true solution is smooth.

This procedure is somewhat impractical for nonlinear shock problems, when spontaneous shocks develop, because their true Fourier series is unknown. The research at ICASE of Gottlieb and Lustman, involving fitting in spectral variable space, is designed to overcome this.

5. Stability

In multidimensions, periodic linear hyperbolic system with variable (say t independent, for convenience only) coefficients can be written:

$$\begin{aligned} u_t &= \left(\sum_{\nu=1}^{d} A_\nu(x)\frac{\partial}{\partial x_\nu} + B(x)\right)u \\ &= Lu. \end{aligned} \tag{5.1}$$

The naive (unsmoothed) Fourier-method approach to this problem, would be to solve:

$$\begin{aligned} u_t^h &= \left(\sum_{\nu=1}^{d} A_\nu(x)\; F^{-1}\cdot i\cdot j_\nu + B(x)\; F^{-1}\right)\tilde{u}(j) \\ &= L^h u^h. \end{aligned} \tag{5.2}$$

There is some controversy as to the stability of this method for general variable coefficient problems, [3], [4].

This naive method, however, cannot work for nonlinear problems. As an example, we consider

$$u_t = -\left(\frac{u^2}{2}\right)_x . \tag{5.3}$$

The naive approach would be: Solve

$$\frac{\partial u^h}{\partial t} = \frac{-F^{-1}\cdot i\cdot jF(u^2)}{2} . \tag{5.4}$$

Take, say, 3 grid points (N=1), with initial data

$$u^h(x_{-1},0) = a, \quad u^h(0,\ 0) = 0, \quad u^h(x_1,0) = -a. \tag{5.5}$$

Then

$$u^h\ (x,0) = \frac{-2a}{\sqrt{3}} \sin x$$

$$\left(u^2(x,0)\right)^h = \frac{2a^2}{3}(1-\cos\ x).$$

The solution to (5.4) is of the form:

$$u^h(x_{-1},t) = a(t), \quad u^h(0,t) = 0, \quad u^h(x_1,t) = -a(t) \tag{5.6}$$

with

$$\frac{da}{dt} = \frac{a^2}{2\sqrt{3}} .$$

Thus $a(t)$ becomes infinite in finite time, and this method is impractical.

Another approach might be to stabilize, using smooth cutoff filters, e.g.

$$\frac{\partial u^h}{\partial t} = -F^{-1}\cdot i\cdot j\rho(jh)F\left(\frac{u^2}{2}\right). \tag{5.7}$$

This method would give the same solution if the initial data is either an expansion shock or a true shock, both moving at zero speed. This means $u^h(x,0) = \text{sgn } x$, gives the same solution as $u^h(x,0) = -\text{sgn } x$.

Thus, a method which approximates u^2, using $(F^{-1}\rho(jh)Fu)^2$ i.e., one which "sees" u, not just u^2, must be used.

For linear hyperbolic problems, to assure stability, we use

$$u_t^h = \left(\sum_{\nu=1}^{d} A_\nu(x)\, F^{-1} \cdot i \cdot j_\nu + B(x)\, F^{-1} \right) \rho(jh)\, \tilde{u}(j), \tag{5.8}$$

where ρ is the usual type of cutoff function. If we solve (5.8) with initial data $u^h(x,0)$ we have in [1], a very general stability theorem (see [1] for precise details).

<u>Theorem 1</u>.

$$\| u^h(\cdot,t) \|_s \leq C_s\, e^{Kt} \| u^h(\cdot,t) \|_s .$$

and

<u>Corollary</u>. If $u(x,0)$ is smooth, then we have infinite order accuracy, globally, for (5.4).

6. Convergence

We recommend solving our stabilized Fourier method (5.4), with the drastically smoothed initial data

$$(6.1) \qquad u^h(x,0) = \sum_{j_d=-N}^{N} \cdots \sum_{j_1=-N}^{N} \sigma(jh)\hat{\phi}(j)e^{ij\cdot x},$$

where $\rho(x)\ \sigma(x) \equiv \sigma(x)$.

A very general convergence theorem was proven in [1]. In a region R_δ, which is essentially any region of smoothness of the true solution to (5.1), we have:

<u>Theorem 2</u>. For any $\lambda > 0$ and any s, $|s| > 0$ there is a constant $C_{\delta,s,\lambda}$ such that

$$\sup_{(x,t)\,\varepsilon\,R_\delta} |\partial_x^s(u-u^h)| < C_{\delta,s,\lambda}h^\lambda.$$

Thus, this method is truly infinitely order accurate. The precise details are given in [1].

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